

# Maps Between Deformed and Ordinary Gauge Fields

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In this paper, we introduce a map between the  $q$ -deformed gauge fields defined on the  $GL_q(N)$ -covariant quantum hyperplane and the ordinary gauge fields. Perturbative analysis of the  $q$ -deformed QED at the classical level is presented and gauge fixing à la BRST is discussed. Another star product defined on the hybrid  $(q, \hbar)$ -plane is explicitly constructed.

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**KEY WORDS:** quantum groups;  $q$ -gauge theories; Seiberg–Witten map; Gerstenhaber star product; Jordanian deformation.

## 1. INTRODUCTION

Motivated by the need to control the divergences which occur in quantum electrodynamics, Snyder (1947a,b) proposed that one may use a noncommutative structure of spacetime coordinates. Although its great success, this suggestion has been swiftly forsaken. This is partly due to a growing development in the renormalization program which captivated all the attention of the leading physicists. The renormalization prescription solved the quantum inconsistencies without making any ad hoc assumption on the spacetime structure. Thanks to the seminal paper of Connes (1985) the interest in noncommutativity (Connes, 1994; Gracia-Bondía *et al.*, 2000; Landi, 1997; Madore, 1999) has been revived. Natural candidates for noncommutativity are provided by quantum groups (Aschieri and Castellani, 1993; Chaichian and Demichev, 1996; Chari and Pressley, 1994; Dobrev, 1993; Drinfeld, 1986; Faddeev *et al.*, 1996; Jimbo, 1986; Kassel, 1995; Klimyk and Schmüdigen, 1997; Manin, 1988) that play the role of symmetry groups in quantum gauge theories (Aref'eva and Volovich, 1991; Bernard, 1990; Brzezinski and Majid, 1993a,b, 1995; Castellani, 1994; Chau and Yamanaka, 1993; Finkelstein,

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2000, 2004; Frishman *et al.*, 1993; Hirayama, 1992; Hou and Ma, 1995; Isaev and Ogievesky, 2001a,b; Isaev and Popowicz, 1993; Mesref, 2002, 2003a,b; Sudbery, 1996a,b,c; Watamura, 1993; Wu and Zhang, 1992) (for a recent review see Ref. Mesref).

In Mesref (2003c), we have constructed a new map which relates a  $q$ -deformed gauge field defined on the Manin plane  $\hat{x}\hat{y} = q\hat{y}\hat{x}$  and the ordinary gauge field. This map is the  $q$ -deformed analogue of the Seiberg–Witten map (Seiberg and Witten, 1999). We have found this map using the Gerstenhaber star product (Gerstenhaber, 1964) instead of the Groenewold–Moyal star product (Groenowold, 1946; Moyal, 1949). In the present paper, we extend our analysis to the general  $GL_q(N)$ -covariant quantum hyperplane defined by  $\hat{x}^i\hat{x}^j = q\hat{x}^j\hat{x}^i \quad i < j$  and to the hybrid plane defined by  $\hat{x}\hat{y} - q\hat{y}\hat{x} = h\hat{y}^2$ .

## 2. $q$ -DEFORMED GAUGE SYMMETRY VERSUS ORDINARY GAUGE SYMMETRY

To begin we consider the undeformed action

$$S = \int d^4x [\bar{\psi} (i\gamma^\mu D_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}], \tag{1}$$

where

$$\begin{aligned} D_\mu \psi &= (\partial_\mu - iA_\mu)\psi \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{2}$$

$S$  is invariant with respect to infinitesimal gauge transformations:

$$\begin{aligned} \delta_\lambda A_\mu &= \partial_\mu \lambda \\ \delta_\lambda \psi &= i\lambda \psi \\ \delta_\lambda \bar{\psi} &= -i\bar{\psi} \lambda. \end{aligned} \tag{3}$$

Now let us study the quantum gauge theory on the quantum hyperplane  $\hat{x}^i\hat{x}^j = q\hat{x}^j\hat{x}^i \quad i < j \quad q \in \mathbb{C}$ .

In general, the product of functions on a deformed space is defined via the Gerstenhaber star product (Gerstenhaber, 1964): Let  $\mathcal{A}$  be an associative algebra and let  $D_i, E^i : \mathcal{A} \rightarrow \mathcal{A}$  be a pairwise commuting derivations.

Then the star product of  $a$  and  $b$  is given by

$$a \star b = \mu \circ e^{\zeta \sum_i D_i \otimes E^i} (a \otimes b), \tag{4}$$

where  $\zeta$  is a parameter and  $\mu$  the undeformed product given by

$$\mu (f \otimes g) = fg. \tag{5}$$

On the quantum hyperplane  $\hat{x}^i \hat{x}^j = q \hat{x}^j \hat{x}^i \quad i < j$ , we can write this star product as:

$$f \star g = \mu \circ e^{i\eta/2(x^i(\partial/\partial x^i) \otimes x^j(\partial/\partial x^j) - x^j(\partial/\partial x^j) \otimes x^i(\partial/\partial x^i))} (f \otimes g). \tag{6}$$

Let us note that the coordinates  $x^i$  are commuting variables, while the quantum coordinates  $\hat{x}^i$  are noncommuting variables. The noncommutative algebra on the quantum hyperplane can be realized on the algebra of the ordinary plane by using the Gerstenhaber star product.

A straightforward computation gives then the following commutation relations

$$x^i \star x^j = e^{i\eta/2} x^i x^j, \quad x^j \star x^i = e^{-i\eta/2} x^j x^i. \tag{7}$$

Whence

$$x^i \star x^j = e^{i\eta} x^j \star x^i, \quad q = e^{i\eta}. \tag{8}$$

Thus, we recover the commutation relations for the quantum hyperplane:  $\hat{x}^i \hat{x}^j = q \hat{x}^j \hat{x}^i$ .

We can also write the product of functions as

$$f \star g = f e^{i/2 \overleftarrow{\partial}_{x^k} \theta^{kl}(x) \overrightarrow{\partial}_l} g \tag{9}$$

where  $\theta^{kl}(x)$  is an antisymmetric matrix depending on the coordinates.

Expanding to first nontrivial order in  $\eta$ , we find

$$\begin{aligned} f \star g &= fg + \frac{i}{2} \eta x^i x^j \left( \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} - \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^i} \right) + o(\eta^2) \quad i < j \\ &= fg + \frac{i}{2} \theta^{ij}(x) \partial_i f \partial_j g + o(\eta^2). \end{aligned} \tag{10}$$

The  $q$ -deformed infinitesimal gauge transformations are defined by

$$\begin{aligned} \widehat{\delta}_{\widehat{\lambda}} \widehat{A}_\mu &= \partial_\mu \widehat{\lambda} + i[\widehat{\lambda}, \widehat{A}_\mu]_\star = \partial_\mu \widehat{\lambda} + i\widehat{\lambda} \star \widehat{A}_\mu - i\widehat{A}_\mu \star \widehat{\lambda}, \\ \widehat{\delta}_{\widehat{\lambda}} \widehat{\psi} &= i\widehat{\lambda} \star \widehat{\psi}, \\ \widehat{\delta}_{\widehat{\lambda}} \widehat{\bar{\psi}} &= -i\widehat{\bar{\psi}} \star \widehat{\lambda}, \\ \widehat{\delta}_{\widehat{\lambda}} \widehat{F}_{\mu\nu} &= i\widehat{\lambda} \star \widehat{F}_{\mu\nu} - i\widehat{F}_{\mu\nu} \star \widehat{\lambda}. \end{aligned} \tag{11}$$

To first order in  $\eta$ , the above formulas for the gauge transformations read

$$\begin{aligned} \widehat{\delta}_{\widehat{\lambda}} \widehat{A}_\mu &= \partial_\mu \widehat{\lambda} - \theta^{\rho\sigma}(x) \partial_\rho \widehat{\lambda} \partial_\sigma \widehat{A}_\mu + o(\eta^2), \\ \widehat{\delta}_{\widehat{\lambda}} \widehat{\psi} &= i\widehat{\lambda} \widehat{\psi} - \theta^{\rho\sigma}(x) \partial_\rho \widehat{\lambda} \partial_\sigma \widehat{\psi} + o(\eta^2), \\ \widehat{\delta}_{\widehat{\lambda}} \widehat{\bar{\psi}} &= -i\widehat{\bar{\psi}} \widehat{\lambda} + \theta^{\rho\sigma}(x) \partial_\rho \widehat{\bar{\psi}} \partial_\sigma \widehat{\lambda} + o(\eta^2), \\ \widehat{\delta}_{\widehat{\lambda}} \widehat{F}_{\mu\nu} &= -\theta^{\rho\sigma}(x) \partial_\rho \widehat{\lambda} \partial_\sigma \widehat{F}_{\mu\nu} + o(\eta^2). \end{aligned} \tag{12}$$

To ensure that an ordinary gauge transformation of  $A$  by  $\lambda$  is equivalent to  $q$ -deformed gauge transformation of  $\widehat{A}$  by  $\widehat{\lambda}$  we consider the following relation (Seiberg and Witten, 1999)

$$\widehat{A}(A) + \widehat{\delta}_{\widehat{\lambda}} \widehat{A}(A) = \widehat{A}(A + \delta_{\lambda} A). \tag{13}$$

We first work the first order in  $\theta$

$$\begin{aligned} \widehat{A} &= A + A'(A) \\ \widehat{\lambda}(\lambda, A) &= \lambda + \lambda'(\lambda, A). \end{aligned} \tag{14}$$

Expanding in powers of  $\theta$  we find

$$A'_{\mu}(A + \delta_{\lambda} A) - A'_{\mu}(A) - \partial_{\mu} \lambda' = \theta^{kl}(x) \partial_k A_{\mu} \partial_l \lambda. \tag{15}$$

The solutions are given by

$$\widehat{A}_{\mu} = A_{\mu} - \frac{1}{2} \theta^{\rho\sigma}(x) (A_{\rho} F_{\sigma\mu} + A_{\rho} \partial_{\sigma} A_{\mu}), \tag{16}$$

$$\widehat{\lambda} = \lambda + \frac{1}{2} \theta^{\rho\sigma}(x) A_{\sigma} \partial_{\rho} \lambda$$

$$\widehat{\psi} = \psi + \frac{1}{2} \theta^{\rho\sigma}(x) A_{\sigma} \partial_{\rho} \psi. \tag{17}$$

The  $q$ -deformed curvature  $\widehat{F}_{\mu\nu}$  is given by

$$\begin{aligned} \widehat{F}_{\mu\nu} &= \partial_{\mu} \widehat{A}_{\nu} - \partial_{\nu} \widehat{A}_{\mu} - i[\widehat{A}_{\mu}, \widehat{A}_{\nu}]_{\star} \\ &= \partial_{\mu} \widehat{A}_{\nu} - \partial_{\nu} \widehat{A}_{\mu} - i \widehat{A}_{\mu} \star \widehat{A}_{\nu} + i \widehat{A}_{\nu} \star \widehat{A}_{\mu}. \end{aligned} \tag{18}$$

Finally, we find

$$\begin{aligned} \widehat{F}_{\mu\nu} &= F_{\mu\nu} + \theta^{\rho\sigma}(x) (F_{\mu\rho} F_{\nu\sigma} - A_{\rho} \partial_{\sigma} F_{\mu\nu}) \\ &\quad - \frac{1}{2} \partial_{\mu} \theta^{\rho\sigma}(x) (A_{\rho} F_{\sigma\nu} + A_{\rho} \partial_{\sigma} A_{\nu}) \\ &\quad + \frac{1}{2} \partial_{\nu} \theta^{\rho\sigma}(x) (A_{\rho} F_{\sigma\mu} + A_{\rho} \partial_{\sigma} A_{\mu}). \end{aligned} \tag{19}$$

From this equation we can see the appearance of terms proportional to  $\partial_{\mu} \theta^{\rho\sigma}(x)$ . Equation (19) can also be written as

$$\widehat{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu} + o(\eta^2), \tag{20}$$

where  $f_{\mu\nu}$  is the quantum correction linear in  $\eta$ . The quantum analogue of Eq. (1) is given by

$$\widehat{S} = \int d^4x \left[ \widehat{\psi} \star (i\gamma^{\mu} \widehat{D}_{\mu} - m) \widehat{\psi} - \frac{1}{4} \widehat{F}_{\mu\nu} \star \widehat{F}^{\mu\nu} \right], \tag{21}$$

where  $\hat{D}_\mu \hat{\psi} = \partial_\mu \hat{\psi} - i \hat{A}_\mu \star \hat{\psi}$ .

We can easily see from this equation that the  $q$ -deformed action contains non-renormalizable vertices of dimension six. Other terms which are proportional to  $\partial_\mu \theta^{\rho\sigma}(x)$  appear.

A gauge fixing term is needed in order to quantize the system. This is done in the BRST and anti-BRST formalism. As usual, the BRST transformations are obtained by replacing  $\hat{\lambda}$  by  $\hat{c}$  and are given by:

$$\begin{aligned} \hat{s} \hat{A}_\mu &= \partial_\mu \hat{c} - \theta^{\rho\sigma}(x) \partial_\rho \hat{c} \partial_\sigma \hat{A}_\mu + \mathcal{O}(\eta^2), \\ \hat{s} \hat{\psi} &= i \hat{c} \hat{\psi} - \theta^{\rho\sigma}(x) \partial_\rho \hat{c} \partial_\sigma \hat{\psi} + \mathcal{O}(\eta^2), \\ \hat{s} \hat{\bar{\psi}} &= -i \hat{\bar{\psi}} \hat{c} + \theta^{\rho\sigma}(x) \partial_\rho \hat{\bar{\psi}} \partial_\sigma \hat{c} + \mathcal{O}(\eta^2), \\ \hat{s} \hat{F}_{\mu\nu} &= -\theta^{\rho\sigma}(x) \partial_\rho \hat{c} \partial_\sigma \hat{F}_{\mu\nu} + \mathcal{O}(\eta^2), \\ \hat{s} \hat{c} &= b, \quad \hat{s} \hat{c} = 0, \quad \hat{s} \hat{b} = 0, \end{aligned} \tag{22}$$

where  $\hat{c}, \hat{\bar{c}}$  are the quantum Faddeev–Popov ghost and anti-ghost fields,  $\hat{b}$  a scalar field (sometimes called the Nielsen–Lautrup auxiliary field) and  $\hat{s}$  the quantum BRST operator. The gauge-fixing term is introduced as

$$\hat{S}_{\text{gf}} + \int d^4x \hat{s} \left( \hat{c} \star \left( \frac{\alpha}{2} \hat{b} - \partial_\mu \hat{A}^\mu \right) \right). \tag{23}$$

An expansion in  $\eta$  leads to an action corresponding to a highly nonlinear gauge.

The external field contribution is given by

$$\hat{S}_{\text{ext}} = \int d^4x \left( \hat{A}^{*\mu} \star \hat{s} \hat{A}_\mu + \hat{c}^* \star \hat{s} \hat{c} \right), \tag{24}$$

where  $\hat{A}^*, \hat{c}^*$  are external fields (called antifields in the Batalin–Vilkovisky formalism) and play the role of sources for the BRST-variations of the fields  $\hat{A}, \hat{c}$ .

The  $\hat{c}$  and  $\hat{\bar{c}}$  play quite asymmetric roles, they cannot be related by Hermitian conjugation. The anti-BRST transformations are given by

$$\begin{aligned} \hat{\bar{s}} \hat{A}_\mu &= \partial_\mu \hat{\bar{c}} - \theta^{\rho\sigma}(x) \partial_\rho \hat{\bar{c}} \partial_\sigma \hat{A}_\mu, \\ \hat{\bar{s}} \hat{\psi} &= i \hat{\bar{c}} \hat{\psi} - \theta^{\rho\sigma}(x) \partial_\rho \hat{\bar{c}} \partial_\sigma \hat{\psi} + \mathcal{O}(\eta^2), \\ \hat{\bar{s}} \hat{\bar{\psi}} &= -i \hat{\bar{\psi}} \hat{\bar{c}} + \theta^{\rho\sigma}(x) \partial_\rho \hat{\bar{\psi}} \partial_\sigma \hat{\bar{c}} + \mathcal{O}(\eta^2), \\ \hat{\bar{s}} \hat{F}_{\mu\nu} &= -\theta^{\rho\sigma}(x) \partial_\rho \hat{\bar{c}} \partial_\sigma \hat{F}_{\mu\nu} + \mathcal{O}(\eta^2), \\ \hat{\bar{s}} \hat{\bar{c}} &= 0, \quad \hat{\bar{s}} \hat{c} = -b, \quad \hat{\bar{s}} \hat{b} = 0. \end{aligned} \tag{25}$$

Here  $\widehat{s}$  is the quantum anti-BRST operator. The complete tree-level action is given by:

$$\sum(\widehat{A}_\mu, \widehat{c}, \widehat{\bar{c}}, \widehat{b}, \widehat{A}^*_\mu, \widehat{c}^*) = \widehat{S} + \widehat{S}_{gf} + \widehat{S}_{ext}. \tag{26}$$

If we replace the ordinary fields and the ordinary action by their  $q$ -deformed analogues we can construct a  $q$ -deformed partition function. This enables us to study the  $q$ -perturbative theory, find the  $q$ -deformed  $n$ -point correlation functions and defined the  $q$ -deformed analogue of the Slavnov–Taylor identity.

### 3. ( $q, h$ )-DEFORMED GAUGE SYMMETRY VERSUS ORDINARY GAUGE SYMMETRY

It is well known (Aghamohammadi, 1993; Aghamohammadi *et al.*, 1995; Demidov *et al.*, 1990; Dubois-Violette and Launer, 1990; Karimipour, 1994, 1995; Kupershmidt, 1992; Ohn, 1992; Vladimirov, 1993; Zakerzewski, 1991) that the only quantum groups that preserve nondegenerate bilinear forms are  $GL_{q,q}(2)$  and  $GL_{h,h}(2)$ . They act on the  $q$ -plane (Manin plane) defined by  $\widehat{X}\widehat{Y} = q\widehat{Y}\widehat{X}$  and on the  $h$ -plane (Jordanian plane) defined by  $\widehat{x}\widehat{y} - \widehat{y}\widehat{x} = h\widehat{y}^2$ , respectively.

In this section, we give the form of the star product of functions defined on the hybrid plane ( $q, h$ )-plane defined by  $\widehat{x}\widehat{y} - q\widehat{y}\widehat{x} = h\widehat{y}^2$ . This map can be used to relate the ( $q, h$ )-gauge fields to the ordinary ones.

Let us recall that the Manin plane and the Jordanian plane are related by a transformation (Aghamohammadi, 1993; Aghamohammadi *et al.*, 1995)

$$\begin{aligned} \begin{pmatrix} \widehat{X} \\ \widehat{Y} \end{pmatrix} &= \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix}, \\ \begin{pmatrix} \partial_{\widehat{X}} \\ \partial_{\widehat{Y}} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -\alpha & 1 \end{pmatrix} \begin{pmatrix} \partial_{\widehat{x}} \\ \partial_{\widehat{y}} \end{pmatrix}, \end{aligned} \tag{27}$$

where  $\alpha = \frac{h}{q-1}$ .

The star product of functions on the Manin plane is defined by choosing the pairwise commuting derivations:  $X \frac{\partial}{\partial X}$  and  $Y \frac{\partial}{\partial Y}$ .

$$f \star g = \mu \circ e^{i\eta/2(X(\partial/\partial X) \otimes Y(\partial/\partial Y) - Y(\partial/\partial Y) \otimes X(\partial/\partial X))} (f \otimes g). \tag{28}$$

A straightforward computation gives the following commutation relations

$$X \star Y = e^{i\eta/2} XY, \quad Y \star X = e^{-i\eta/2} YX \tag{29}$$

Whence

$$X \star Y = e^{i\eta} Y \star X, \quad q = e^{i\eta}. \tag{30}$$

Thus, we recover the commutation relations for the Manin plane.

On the hybrid space, we define the star product as

$$f \star g = \mu \circ e^{i\eta/2[(x(\partial/\partial x) + \alpha y(\partial/\partial x)) \otimes (y(\partial/\partial y) - \alpha y(\partial/\partial x)) - (y(\partial/\partial y) - \alpha y(\partial/\partial x)) \otimes (x(\partial/\partial x) + \alpha y(\partial/\partial x))]} (f \otimes g). \tag{31}$$

A direct computation gives

$$x \star y = e^{i\eta/2}xy + (e^{i\eta/2} - 1)\alpha y^2, \quad y \star x = e^{-i\eta/2}yx + (e^{-i\eta/2} - 1)\alpha y^2. \tag{32}$$

Whence

$$\begin{aligned} x \star y &= e^{i\eta}y \star x + (e^{i\eta} - 1)\alpha y^2 \\ &= q y \star x + h y^2. \end{aligned} \tag{33}$$

Thus, we recover the commutation relations for the hybrid plane.

Expanding to first nontrivial order in  $\eta$  and  $h$  we find

$$f \star g = f g + \frac{i}{2}(\eta xy - h y^2) \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right). \tag{34}$$

If we take  $\eta = h = 0$  we recover the ordinary product of commuting functions defined on the ordinary two-dimensional plane.

We can also write the star product (defined by Eq. (31) ) as

$$f \star g = f e^{i/2 \overleftarrow{\partial}_k \Theta^{kl}(x,y) \overrightarrow{\partial}_l} g. \tag{35}$$

Here, the antisymmetric matrix  $\Theta^{kl}(x, y) = (\eta xy - h y^2)\epsilon^{kl}$  with  $\epsilon^{12} = \epsilon^{21} = 1$ .

This star product can be used to relate the  $(q, h)$ -deformed gauge fields to the ordinary ones.

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**REFERENCES**

Aghamohammadi, A. (1993). *Modern Physics Letters A* **8**, 2607.  
 Aghamohammadi, A., Rouhani, S., and Shariati, A. (1995). *Journal of Physics A* **28**, L225.  
 Aref’eva, I. Ya. and Volovich, I. V. (1991). *Modern Physics Letters A* **6**, 893.  
 Aschieri, P. and Castellani, L. (1993). *International Journal of Modern Physics A* **8**, 1667.  
 Bernard, D. (1990). *Progress of Theoretical Physics Supplement* **102**, 49.

- Bo-Yan Hou and Zhong-Qi Ma (1995). *Journal of Mathematical Physics* **36**, 5110.
- Brzezinski, T. and Majid, S. (1993). *Communications in Mathematical Physics* **157**, 591.
- Brzezinski, T. and Majid, S. (1993). *Physics Letters B* **298**, 339.
- Brzezinski, T. and Majid, S. (1995). **167**, 235 [Erratum].
- Castellani, L. (1994). *Physics Letters B* **237**, 22.
- Chaichian, M. and Demichev, A. P. (1996). *Introduction to Quantum Groups* World Scientific.
- Chari, V. and Pressley, A. N. (1994). *Guide to Quantum Groups*, Cambridge University Press, Cambridge.
- Chau, L. L. and Yamanaka, I. (1993). *Physical Review Letters* **70**, 1916.
- Connes, A. (1985). *Publications Mathématiques. Institut des Hautes Études Scientifiques* **62**, 257.
- Connes, A. (1994). *Noncommutative Geometry*, Academic Press, San Diego, California.
- Demidov, E. E., Manin, Yu. I., Mukhin, E. E., and Zhdanovich, D. V. (1990). *Progress of Theoretical Physics Supplement* **102**, 203.
- Dobrev, V. K. (1993). *Journal of Geometry and Physics* **11**, 367.
- Drinfeld, V. G. (1986). In *Proceedings of the International Congress of Mathematicians*, Berkeley, California, A. M. Gleason, ed., American Mathematical Society, Providence, RI, p. 798.
- Dubois-Violette, M. and Launer, G. (1990). *Physics Letters B* **245**, 175.
- Faddeev, L. D., Reshetikhin, N., and Pyatov, P. N. (1996). *American Mathematical Society Translations* **175**, 35.
- Finkelstein, R. J. (2000). *Modern Physics Letters* **15**, 1709.
- Finkelstein, R. J. (2004). *International Journal of Modern Physics* **19**, 443.
- Frishman, Y., Lukierski, J., and Zakrzewski, W. J. (1993). *Journal of Mathematics A* **26**, 301.
- Gerstenhaber, M. (1964). *Annals of Mathematics* **79**, 59.
- Gracia-Bondía, J. M., Várilly, J. C., and Figueroa, H. (2000). *Elements of Noncommutative Geometry*, Birkhäuser.
- Groenowold, H. J. (1946). On the principles of Elementary Quantum Mechanics. *Physica* **12**, 405.
- Hirayama, M. (1992). *Progress of Theoretical Physics* **88**, 111.
- Isaev, A. P. and Popowicz, Z. (1993). *Physics Letters B* **307**, 353.
- Isaev, P. and Ogievesky, O. V. (2001a). *Nuclear Physics Proceedings Supplement* **102**, 306.
- Isaev, P. and Ogievesky, O. V. (2001b). *Theoretical and Mathematical Physics* **129**, 1558.
- Jimbo, M. (1986). *Letters in Mathematical Physics* **10**, 63.
- Karimipour, V. (1994). *Letters in Mathematical Physics* **30**, 87.
- Karimipour, V. (1995). *Letters in Mathematical Physics* **35**, 303.
- Kassel, C. (1995). *Quantum Groups*, Graduate Texts in Mathematics, Vol. 155, Springer-Verlag, Berlin.
- Klimyk, A. and Schmüdigen, K. (1997). *Quantum Groups and Their Representations*, Springer-Verlag, Berlin.
- Kupershmidt, B. A. (1992). *Journal of Physics A* **25**, L1239.
- Landi, G. (1997). *An Introduction to Noncommutative Spaces and their Geometries*, Springer-Verlag, Berlin.
- Madore, J. (1999). *An Introduction to Noncommutative Geometry and its Physical Applications*, Cambridge University Press, Cambridge.
- Manin, Yu. I. (1988). *Quantum Groups and Noncommutative Geometry*, Montreal University preprint, CRM-1561.
- Mesref, L. (2002). *International Journal of Modern Physics A* **17**, 4777.
- Mesref, L. (2003a). *International Journal of Modern Physics A* **18**, 209.
- Mesref, L. (2003b). *Physical Review D* **68**, 065007.
- Mesref, L. (2003e). *New Journal of Physics* **5**, 7.
- Mesref, L. *Quantum Gauge Theories* (hep-th/0412158).
- Moyal, J. E. (1949). Quantum mechanics as a statistical theory. *Proceedings of the Cambridge Philological Society* **45**, 99.



- Ohn, Ch. (1992). *Letters in Mathematical Physics* **25**, 85.
- Seiberg, N. and Witten, E. (1999). *JHEP* **9909**, 032.
- Snyder, H. S. (1947a). Quantized spacetime. *Physical Review* **71**, 38.
- Snyder, H. S. (1947b). The electromagnetic field in quantized spacetime. *Physical Review* **72**, 68.
- Sudbery, A. (1996a). *Physics Letters B* **375**, 75.
- Sudbery, A. (1996b). *Acta Phys. Polon.* **B27**, 2777.
- Sudbery, A. (1996c). Talk prepared for *21st International Colloquium on Group Theoretical Methods in Physics* (ICGTMP 96), Goslar, Germany, 16–20 July 1996 (In Goslar (1996). *Quantum group*, 45–52).
- Vladimirov, A. A. (1993). *Modern Physics Letters A* **8**, 2573.
- Watamura, S. (1993). *Communications in Mathematical Physics* **158**, 67.
- Wu, K. and Zhang, R. J. (1992). *Communications in Theoretical Physics* **17**, 175.
- Zakerzewski, S. (1991). *Letters in Mathematical Physics* **22**, 287.